MATHEMATICS

BY

Dr. Mohammed Jasim Mohammed

Aflahdawi

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

2019-2020

Solution Set of Inequalities

The solution set of on inquality consists of the set real numbers for which the inquality is true state ment if two inequalities have the same solution set, they are said to be equivalent.

Example: Find the solution set of the following inequalities.

$$3x-8<x-2$$
$$3x-8+8<x-2+8$$
$$3x<x+6$$
$$3x-x<x-x+6$$
$$2x<6$$
$$2x.\frac{1}{2}<6.\frac{1}{2}$$

$$x<3 \Rightarrow \delta=\{x\in \mathbb{R}, -\infty<x<3\}=(-\infty,3)$$

Example: Find the solution set of the following inequalities.

$$\frac{2x-3}{x+2}<\frac{1}{3} \qquad , x\neq -2$$

$$if \quad x+2>0 \Rightarrow 3(2x-3)<x+2$$
$$6x-9<x+2$$
$$5x<11$$
$$x<\frac{11}{5}$$

$$\delta=\left\{x\in\mathbb{R}\ ;\ x<\frac{11}{5}\ \text{ and }\ x>-2\right\}$$

$$=\left\{x\in\mathbb{R}\ ;x<\frac{11}{5}\right\}\bigcap\left\{x\in\mathbb{R}\ ;x>-2\right\}=(-2,\frac{11}{5}\}$$

$$if\quad x+2<0\Rightarrow x<-2$$

$$3(2x-3)>x+2$$

$$6x-9>x+2$$

$$5x>11\Rightarrow x>\frac{11}{5}$$

$$\delta=\left\{x\in\mathbb{R}\ ;x>\frac{11}{5}\right\}\bigcap\left\{x\in\mathbb{R}\ ;x<-2\right\}\ =\Phi$$

<u>Example:</u> Find the solution set of the following inequalities.

$$x^2-3x+2<0$$

$$(x-2)(x-1)<0$$

$$x-2<0\quad\text{and}\quad x-1>0$$

$$or$$

$$x-2>0\quad\text{and}\quad x-1<0$$

$$x-2<0\Rightarrow x<2\ \text{ and }\ x-1>0\quad x>1$$

$$\delta=\left\{x\in\mathbb{R}\ ;\ \ x<2\right\}\bigcap\left\{x\in\mathbb{R}\ ;\ \ x>1\right\}=(1,2\}$$

$$x-2>0\Rightarrow x>2\ \text{ and }\ x-1<0\quad x<1$$

$$\delta=\left\{x\in\mathbb{R}\ ;\ \ x>2\right\}\bigcap\left\{x\in\mathbb{R}\ ;\ \ x<1\right\}=\Phi$$

Theorem of Limit

Not the following rules hold if

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M$$

1- $\lim_{x \to a} C = C \quad where \quad C \in R$

2- $\lim_{x \to a} Cf(x) = C \lim_{x \to a} f(x) = CL \quad where \quad C \in R$

3- $\lim_{x \to a} (f(x) \mp g(x)) = \lim_{x \to a} f(x) \mp \lim_{x \to a} g(x) = L \mp M$

4- $\lim_{x \to a} (f(x).g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = LM$

5- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \quad , where \;\; M \neq 0$

6- $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = L^n$

Example: Evaluate the following Limits

$$\lim_{x\to 0}\left[\frac{x^4-x+1}{x-1}\right]^3=\left[\lim_{x\to 0}\frac{x^4-x+1}{x-1}\right]^3$$

$$=\left[\frac{0-0+1}{0-1}\right]^3=-1$$

Example:

$$\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\left[\frac{x^2-25}{x-5}\right]$$

Sol/

$$\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\lim_{x\to 5}\left[\frac{x^2-25}{x-5}\right]$$

$$\lim_{x\to 5}\left[\frac{(x-5)(x+5)}{x+5}\right]\lim_{x\to 5}\left[\frac{(x-5)(x+5)}{x-5}\right]$$

$$=(5-5)(5+5)$$

$$=(0)(10)=0$$

Example:

$$\lim_{y\to 2}\frac{\sqrt{y^2+12}-4}{y-2}$$

Sol/

$$=\lim_{y\to 2}\frac{\sqrt{y^2+12}-4}{y-2}\cdot\frac{\sqrt{y^2+12}+4}{\sqrt{y^2+12}+4}$$

$$=\lim_{y\to 2}\frac{y^2+12-16}{y-2\left(\sqrt{y^2+12}+4\right)}$$

$$=\lim_{y\to 2}\frac{y^2-4}{y-2\left(\sqrt{y^2+12}+4\right)}$$

$$=\lim_{y\to 2}\frac{(y-2)(y+2)}{(y-2)\left(\sqrt{y^2+12}+4\right)}$$

$$=\frac{2+2}{4+4}=\frac{1}{2}$$

H.W

$$\lim_{t\to 2}\frac{t-4}{t^2-t-12}$$

$$\lim_{x\to -1}\frac{x^3+x+2}{x+1}$$

Limits and Continuity of Trigonometric functions

Theorem: If c is any number in the natural domain of the stated trigonometric function, then:

1- $\lim_{x\to c} \sin x = \sin c$
2- $\lim_{x\to c} \cos x = \cos c$
3- $\lim_{x\to c} \tan x = \tan c$
4- $\lim_{x\to c} \csc x = \csc c$
5- $\lim_{x\to c} \sec x = \sec c$
6- $\lim_{x\to c} \cot x = \cot c$

i.e. $\sin x$, $\cos x$ are continuous and so tan x is continuous except at the point where it is undefined .

Exemple: Find $\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right)$

Sol: $let\ f(x) = \cos x$

$$g(x) = \frac{x^2-1}{x-1}$$

$\because \lim_{x\to 1} g(x) = \frac{x^2-1}{x-1}$

$$= \lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1}(x+1) = 2$$

And $f(x) = \cos(x)$ is continuous at 2

$\therefore \lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right) = \cos\left(\frac{x^2-1}{x-1}\right) = \cos 2.$

Theorem:

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

Proof:

$$\lim_{x\to 0}\frac{1-\cos x}{x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x}.\frac{1+\cos x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{sin^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\left(\frac{sin\ x}{x}.\frac{\sin x}{(1+\cos x)}\right)=1.\frac{0}{2}=0$$

Exammple: Find

1- $\lim_{x\to 0}\frac{\tan x}{x}$

Sol

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{\mathrm{x}\cos x}$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right).\left(\lim_{x\to 0}\frac{1}{\cos x}\right)=1.1=1$$

Type equation here.

2- $\lim_{x\to 0}\frac{\sin 2x}{x}$

Sol

$$\lim_{x\to 0}\frac{2\sin 2x}{2x}$$

$$=2.\lim_{x\to 0}\frac{\sin 2x}{2x}=2$$

3- $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$

Sol

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}.\frac{1/x}{1/x}$$

$$=\lim_{x\to 0}\frac{\sin 3x/3x}{\sin 5x/5x}=\frac{3}{5}$$

Infinite Limits

Sometimes we need to know what happens to *f(x)* as *x* gets Large and positive

$(x \to \infty)$ or large and negative$(x \to -\infty)$ Consider a function $f(x) = \frac{1}{x}$ what dose

$\lim f(x)$. *f(x)* gest close to 0, as *x* gets Large and $(x \to \infty)$ large. This is written

$$\lim_{x\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} = \mathbf{0} \quad \boldsymbol{or} \quad \lim_{x\to-\infty} \frac{\mathbf{1}}{\boldsymbol{x}} = \mathbf{0}$$

Example: Find the following Limits if they exist.

$$\lim_{x\to\infty} \frac{x^3 + 2x + 1}{3x^3 + 1}$$

Sol/

$$\lim_{x\to\infty} \frac{\frac{x^3}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{3x^3}{x^3} + \frac{1}{x^3}}$$

$$\lim_{x\to\infty} \frac{1 + \frac{2}{x^2} + \frac{1}{x^3}}{3 + \frac{1}{x^3}}$$

$$= \frac{1}{3}$$

Example:

$$\lim_{x\to\infty}\frac{4x-2}{x^2+3}$$

Sol/

$$=\lim_{x\to\infty}\frac{\frac{4x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\frac{4}{x}-\frac{2}{x^2}}{1+\frac{3}{x^2}}$$

$$\frac{0}{1}=0$$

H.W

$$\lim_{x\to\infty}\sqrt{\frac{9x+1}{x-1}}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$$

$$\lim_{x\to\infty}\frac{\sqrt{x^2+x}}{x+1}$$

$$\lim_{x\to 0}\frac{2x^2+3x}{5x^2-2x}$$

Derivatives

The derivative of the function $y = f(x)$ is the function $y' = f'(x)$ whose value at each x is define by rule.

Let $y = f(x)$ be a function of x . If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

exists and is finite, we call this limit the derivative of f at x and say that f is differentiable at x .

Example: Find the derivative of the function $\boldsymbol{y = x^2}$ by definition

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Sol/

$$f(x+\Delta x) = (x+\Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2$$

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Example: Using the definition of the derivative of a function to the derivative of the functions

$$f(x) = x^3 + 2x$$

Sol/

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2\Delta x + 2x\Delta x^2 + \Delta x^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 2x\Delta x^2 + \Delta x^3 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 2x\Delta x + \Delta x^2 + 2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 2x\Delta x + \Delta x^2 + 2)$$

$$= 3x^2 + 2$$

H.W

Find the derivative of the function by definition.

1- $f(x) = \dfrac{1}{\sqrt{2x+3}}$ **2-** $f(x) = \sqrt{x}$

Derivatives

The derivative of the function $y = f(x)$ is the function $y' = f'(x)$ whose value at each x is define by rule.

The Rules for Derivative

1- $y = a \Rightarrow \frac{dy}{dx} = 0$

Ex/ $y = 2 \Rightarrow \frac{dy}{dx} = 0$

2- $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$

Ex/ $y = x^{-2} \Rightarrow \frac{dy}{dx} = -2x^{-2-1} = -2x^{-3}$

3- $y = ax^n \Rightarrow \frac{dy}{dx} = a.nx^{n-1}$

Ex/ $y = 4\sqrt[3]{x} \Rightarrow \frac{dy}{dx} = 4.\frac{1}{3}x^{\frac{1}{3}-1}$

4- $y = u(x) + v(x) \Longrightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Ex/ $y = 2x^2 + 8 - 5x^4 \Longrightarrow \frac{dy}{dx} = 4x + 0 - 20x^3$

5- $y = b[u(x)]^n \Longrightarrow \frac{dy}{dx} = b.n[u(x)]^{n-1}\frac{du}{dx}$

Ex/ $y = 3(2x^2 - x + 4)^7 \Longrightarrow \frac{dy}{dx} == 3.7(2x^2 - x + 4)^6.(4x - 1)$

6- $y = u(x).v(x) \Rightarrow \frac{dy}{dx} = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$

Ex/ $y = (x^2+1)(x-3)^2 \Rightarrow \frac{dy}{dx} = (x^2+1).2(x-3)^2 + (x-3)^2(2x)$

7- $y = \frac{u(x)}{v(x)} \Rightarrow \frac{dy}{dx} = \frac{v(x).\frac{dv}{dx} - u(x)\frac{du}{dx}}{[v(x)]^2}$

Ex/ $y = \frac{x^2+1}{3x^2+2x} = \frac{dy}{dx} = \frac{(3x^2+2x).(2x)-(x^2+1).(6x+2)}{9x^4+12x^3+4x^2}$

$= \frac{2x^2-6x-2}{9x^4+12x^3+4x^2}$

Example: Find $\frac{dy}{dx}$ for the following functions :

$a)\ y = (x^2+1)^5$

$b)\ y = [(5-x)(4-2x)]^2$

$c)\ y = (2x^3-3x^2+6x)^{-5}$

$d)\ y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

$e)\ y = \frac{(x^2+x)(x^2-x+1)}{x^3}$

$f)\ y = \frac{x^2-1}{x^2+x-2}$

Sol/

$a)\ \frac{dy}{dx} = 5(x^2+1)^4.2x = 10x(x^2+1)^4$

$b)\ \frac{dy}{dx} = 2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$= 8(5-x)(2-x)(2x-7)$

Sol/

$$c)\quad \frac{dy}{dx} = -5(2x^3 - 3x^2 + 6x)^{-6}(6x^2 - 6x + 6)$$
$$= -30(2x^3 - 3x^2 + 6x)^{-6}(x^2 - x + 1)$$

$$d)\quad y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$$

$$e)\quad y = \frac{(x+1)(x^2 - x + 1)}{x^3} \Rightarrow$$
$$\frac{dy}{dx} = \frac{x^3\left[(x^2 - x + 1) + (x+1)(2x-1)\right] - 3x^2(x+1)(x^2 - x + 1)}{x^6} = -\frac{3}{x^4}$$

$$f)\quad \frac{dy}{dx} = \frac{2x(x^2 + x - 2) - (x^2 - 1)(2x+1)}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2}$$

H.W

1- $f(x) = x^3 . \frac{1}{x^2+1}$

3- $y = (2\sqrt{x} - 1)^3$

2- $y = \frac{\sqrt{x^2+1}}{(x+2)^2}$

4- $f(x) = (x^3 + 2)^2(1 - x^2)^3$

Find $f'(x) = y'$

The derivative of trigonometry function

1) $y = \sin\big(g(x)\big) \Rightarrow y' = \cos\big(g(x)\big).g'(x)$
2) $y = \cos\big(g(x)\big) \Rightarrow y' = -\sin\big(g(x)\big).g'(x)$
3) $y = \tan\big(g(x)\big) \Rightarrow y' = sec^2\big(g(x)\big).g'(x)$
4) $y = \cot\big(g(x)\big) \Rightarrow y' = -csc^2\big(g(x)\big).g'(x)$
5) $y = \sec\big(g(x)\big) \Rightarrow y' = \tan\big(g(x)\big).\sec\big(g(x)\big)\, g'(x)$
6) $y = \csc\big(g(x)\big) \Rightarrow y' = -\cot\big(g(x)\big).\csc\big(g(x)\big)\, g'(x)$

Example: Find the derivative of the following function

$$f(x) = sin(x)^2 + \cot(x^4 - 1)$$

Sol/

$$f'^{(x)} = cos(x)^2.2x - \csc^2(x^4 - 1).4x^3$$

Dr. Mohammed Jasim Mohammed Alfahdawi

Example: Find the derivative of the following function

$$f(x) = (\sec(2x) + \tan(3x))^{-2}$$

Sol/

$$f'^{(x)} = -2(\sec(2x) + \tan(3x))^{-3}.(\sec(2x).\tan(2x).2 + 3sec^2(3x))$$

Example: Find the derivative of the following function

$$f(x) = \frac{2}{\cos(3t)}$$

Sol/

$$f'^{(x)} = \frac{\cos(3t).0 - 2.(-\sin(3t).3}{cos^2(3t)} = \frac{6\sin(3t)}{cos^2(3t)}$$

Example: Find the derivative of the following function

$$f(x) = \sin(\cos(w))$$

Sol/

$$f'(x) = \cos(\cos(w)).(-\sin(w)).1$$

H.W

1. $y = \left(\frac{sin\sqrt{x}}{\sqrt{x}}\right)^3$

2. $y = 2sin\left(\frac{z}{2}\right) - \left(x.cos\left(\frac{z}{2}\right)\right)^3$

3. $y = \frac{7\sqrt{\sec(3\theta)}}{\theta^2}$

4. $y = \sin(3t).\cos(5t^2)$

Derivatives

The derivative of the function $y = f(x)$ is the function $y'' = f''(x)$ whose value at each x is define by rule.

Example: Find y''

$$y = x^6 + 3x^4 - 2x^2 + 9$$

Sol/

$$y' = 6x^5 + 12x^3 - 4x$$

$$y'' = 30x^4 + 36x^2 - 4$$

Derivatives

The derivative of the function $y = f(x)$ is the function $y' = f'(x)$ whose value at each x, y is define by rule.

Example: Find y' of the following functions

$$x.\sin(2y) = y.\cos(2x)$$

Sol/

$x.\cos(2y)\,.\,2y' + \sin(2y)\,.\,1 = y.(-\sin(2x)).\,2 + \cos(2x)\,.\,y'$

$x.\cos(2y)\,.\,2y' - \cos(2x)\,.\,y' = y.(-\sin(2x)).\,2 - \sin(2y)$

$x.\cos(2y)\,.\,2y' - \cos(2x)\,.\,y' = y.(-\sin(2x)).\,2 - \sin(2y)$

$(2x.\cos(2y) - \cos(2x)).\,y' = -2y.\sin(2x) - \sin(2y)$

$$y' = \frac{-2y.\sin(2x) - \sin(2y)}{(2x.\cos(2y) - \cos(2x))}$$

Example: Find y' of the following functions

$$\cot(xy) + xy = 0$$

Sol//

$-csc^2(xy)(xy' + y.1) + xy' + y = 0$

$-x.csc^2(xy)y' - ycsc^2(xy) + xy' + y = 0$

$(-x.csc^2(xy) + x)y' = ycsc^2(xy) - y$

$y' = \frac{ycsc^2(xy) - y}{(-x.csc^2(xy) + x)}$

$y' = \frac{y(csc^2(xy) - 1)}{-x(.csc^2(xy) - 1)}$

$y' = \frac{y}{-x}$

H.W

1. $y = \tan y + sec^2(xy) + \cot(x^2 + y^2)$
2. $\cos(x^2y^2) = x$
3. $x^2y = \frac{\cot y}{1+\csc y}$
4. $\sqrt{xy} + \csc(-xy) = y$

The Derivative of Inverse Trigonometric Functions:

Let *u* be a function of *x* , then:

(1) $\frac{d}{du}\boldsymbol{sin^{-1}(u)} = \frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$

(2) $\frac{d}{du}\boldsymbol{cos^{-1}(u)} = -\frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$

(3) $\frac{d}{du}\boldsymbol{tan^{-1}(u)} = \frac{1}{1+u^2}.\frac{du}{dx}$

(4) $\frac{d}{du}\boldsymbol{cot^{-1}(u)} = -\frac{1}{1+u^2}.\frac{du}{dx}$

(5) $\frac{d}{du}\boldsymbol{sec^{-1}(u)} = \frac{1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$

(6) $\frac{d}{du}\boldsymbol{csc^{-1}(u)} = -\frac{1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$

Examples: Find the derivatives of the following functions:

■ $f(x) = sin^{-1}(x^2)$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1-(x^2)^2}}.2x$$

■ $f(t) = cos^{-1}(\sqrt{t})$

$$\Rightarrow f'(t) = \frac{1}{\sqrt{1-\left(\sqrt{t}\right)^2}}.\frac{1}{2}t^{\frac{-1}{2}}$$

Dr. Mohammed Jasim Mohammed Alfahdawi

■ $f(x) = x.csc^{-1}\left(\frac{1}{x}\right) + \sqrt{1+x^2}$

$$\Rightarrow f'(x) = x.\frac{-1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}.\frac{x.0-1.1}{x^2} + csc^{-1}\left(\frac{1}{x}\right).1 + \frac{1}{2}(1-x^2)^{\frac{-1}{2}}.-2x$$

H.W

1- $y = sin^{-1}\frac{x-1}{x+1}$

2- $sec^{-1}\left(\sqrt{w^2+4}\right)$

3- $y = tan^{-1}(3\tan 2z)$

4- $y = \frac{cot^{-1}(3\theta)}{1+\theta^2}$

The Derivative of ln

$$\frac{d}{dx} \ln u = \frac{1}{u} . \frac{du}{dx}$$

Examples: Find y' for the following function:

1. $y = \ln(x^2 + 2x) \Rightarrow y' = \frac{2x + 2}{x^2 + 2x}$

2. $y = (\ln x)^3 \Rightarrow y' = 3\ (\ln x)^2 . \frac{1}{x}$

3. $y = \ln(\ln x) \Rightarrow y' = \frac{\frac{1}{x}}{\ln x}$

H.W

1) $y = (\ln \tan)^5 . \cos x^2$

2) $y = Lnx - \frac{1}{2} \ln(1 + x^2) - \frac{tan^{-1} x}{x}$

The Derivative of e^x

$$\frac{d}{dx}e^u = e^u.\frac{du}{dx}$$

$$1.\ e^0 = 1 \qquad\qquad 2.\ e^{\ln x} = x = \ln e^x$$

Examples: Find y' for the following function:

■ $y = e^{tan^{-1}x} \Rightarrow y' = e^{tan^{-1}x}.\frac{1}{1+x^2}$

■ $y = ln\frac{e^x}{1+e^x}$

$$\Rightarrow y' = \frac{1}{\frac{e^x}{1+e^x}}.\frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2}$$

■ $y = sec^{-1}(e^{2x})$

$$y' = \frac{1}{|e^{2x}|\sqrt{(e^{2x})^2-1}}.e^{2x}.2$$

H.W

1) $y = \frac{e^z - e^{-z}}{e^z + e^z}$ 2) $y = e^{sin^{-1}x}\sqrt{x}$ 3) $y = \ln\left(e^{-\sin\theta} - e^{-\theta^3+5}\right)$

Integration

General Indefinite Integral:

In Calculus, an antiderivative of a function *f(x)* is a differentiable function *F(x)* whose derivative is equal to the original function *f(x)* .

$$F'(x) = f(x) \Longrightarrow \frac{dF(x)}{dx} = f(x)$$

$$\Longrightarrow dF(x) = f(x)dx$$

$$\Longrightarrow \int dF(x) = \int f(x)dx$$

$$\Longrightarrow F(x) = \int f(x)dx + C, \quad \text{where} \quad C \in \mathbb{R}.$$

Example: $y' = 2x \Longrightarrow \frac{dy}{dx} = 2x \Longrightarrow dy = 2x\,dx$

$$\Longrightarrow \int dy = \int 2x\,dx \quad \Longrightarrow y = \frac{x^2}{2} + C$$

Indefinite Integrals Properties: Let f(x) be a function and $x \in \mathbb{R}$, then:

i. $\int af(x) = a\int f(x)\,dx,$ where a is a constant.

ii. $\int (f(x) \mp g(x))\,dx = \int f(x)\,dx \mp \int g(x)\,dx$

Notes:

- $\int (f(x) * g(x))\,dx \neq \int f(x)dx * \int g(x)dx$
- $\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$

Rules: In general,

i. $\int dx = x + c$

ii. $\int u^n du = \frac{u^{n+1}}{n+1} + C$, where $C \in \mathbb{R}$.

Examples: Evaluate the following integrals:

1) $\int 3dx = 3\int dx = 3x + C$

2) $\int 5x^2 dx = 5\int x^2 dx = 5\frac{x^3}{3} + C$

3) $\int 6x^{-3}dx = 6\int x^{-3}dx = 6\frac{x^{-2}}{-2} = \frac{-3}{x^2} + C$

4) $\int \frac{3\pi}{x^5}dx = \int 3\pi\, x^{-5}dx = 3\pi\frac{x^{-4}}{-4} + C = \frac{-3\pi}{4x^4} + C$

5) $\int (2x + 3)\,dx = \int 2xdx + \int 3dx$

$= x^2 + C_1 + 3x + C_2 = x^2 + 3x + C,$ where $C = C_1 + C_2$.

1) $\int \sqrt{2x+1}\,dx = \frac{1}{2}\int (2x+1)^{\frac{1}{2}}.2dx$

$$= \frac{1}{2}\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}\sqrt{(2x+1)^3} + C$$

2) $\int \left(x^2 - \sqrt{x}\right) dx = \int x^2\,dx - \int x^{\frac{1}{2}}\,dx$

$$= \frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

3) $\int \frac{(z+1)dz}{\sqrt[3]{z^2+2z+2}} = \int (z^2+2z+2)^{\frac{-1}{3}}(z+1)dz$

$$= \frac{1}{2}\frac{(z^2+2z+2)^{\frac{2}{3}}}{\frac{2}{3}} + C$$

$$= \frac{3}{4}\sqrt[3]{(z^2+2z+2)^2} + C$$

Dr. Mohammed Jasim Mohammed Alfahdawi

H.W

1. $\int (x^2 + 5x)\, dx$

2. $\int \sqrt{y^2 + 2y}(y + 1)\, dy$

3. $\int \frac{3t}{\sqrt{2-t^2}}\, dt$

4. $\int \frac{dx}{(3x+2)^2}$

5. $\int \frac{-7x}{(x^2-16)^8}\, dx$

6. $\int \sqrt[3]{r^3 + \pi}\, r^2 dr$

Integrals of Trigonometric Function:

We can derive all the Trigonometric integration forms from the derivative

Trigonometric forms as follows:

(1) $\frac{d}{du}\sin(u) = \cos(u) \rightarrow \int \cos(u)\, du = \sin(u) + C$

(2) $\frac{d}{du}\cos(u) = -\sin(u) \rightarrow \int \sin(u)\, du = -\cos(u) + C$

(3) $\frac{d}{du}\tan(u) = sec^2(u) \rightarrow \int sec^2(u)\, du = \tan(u) + C$

(4) $\frac{d}{du}\cot(u) = -csc^2(u) \rightarrow \int csc^2(u)\, du = -\cot(u) + C$

(5) $\frac{d}{du}\sec(u) = \sec(u)\tan(u) \rightarrow \int sec(u)\tan(u) du = \sec(u) + C$

(6) $\frac{d}{du}\csc(u) = -\csc(u)\cot(u) \rightarrow \int csc(u)\cot(u) du = -\csc(u) + C$

Examples: Evaluate the following integrals:

1) $\int \sin(3x)\, dx$

$= \frac{1}{3}\int \sin(3x) . 3dx$

$= -\frac{1}{3}\cos(3x) + C$

2) $\int \cos(2t)\, dt$

$$= \frac{1}{2}\int \cos(2t)\,.\,2dt$$

$$= \frac{1}{2}\sin(2x) + C$$

3) $\int xsec^2(x^2)dx$

$$= \frac{1}{2}\int sec^2(x^2).2xdx$$

$$= \frac{1}{2}\tan(x^2) + C$$

4) $\int cot(5x)\, csc(5x)\, dx$

$$= \frac{1}{5}\int \cot(5x)\csc(5x)\,.\,5dx$$

$$= -\frac{1}{5}\csc(5x) + C$$

5) $\int sec^3(x)tan(x)dx$

$$= \int sec^2(x)\sec(x)\, tan(x)\, dx$$

$$= \frac{sec^3(x)}{3} + C$$

6) $\int \frac{\cos(2x)}{sin^3(2x)} dx$

$= \frac{1}{2} \int (\sin(2x))^{-3} cos92x).2dx$

$= \frac{1}{2} \frac{(\sin(2x))^{-2}}{-2} + 2$

$= -\frac{1}{4sin^2(2x)} + C$

7) $\int \frac{6 - \cos(3x)}{sin^2(3x)} dx$

$= \int \frac{6}{sin^2(3x)} dx - \int \frac{\cos(3x)}{\sin^2(3x)} dx$

$= 6 \int \frac{1}{sin^2(3x)} dx - \int \frac{1}{\sin(3x)} \frac{\cos(3x)}{\sin(3x)} dx$

$= 6 \int csc^2(3x)dx - \int \csc(3x) \cot(3x)\, dx$

$= \frac{6}{3} \int csc^2\,(3x).\,3dx - \frac{1}{3} \int \csc(3x) \cot(3x)\,.\,3dx$

$= 2(-\cot(2x)) - \frac{1}{3}(-\csc(3x)) + C$

$= -2\cot(3x) + \frac{1}{3} \csc(3x) + C$

H.W

1) $\int \frac{1}{\sqrt{x}} csc^2\left(\sqrt{x}\right) dx =$

2) $\int \mathrm{t}^2 \tan^4(\mathrm{t}^3)\mathrm{dt} =$

3) $\int csc^7(x) \cot(x)\, dx =$

4) $\int \frac{sec^2(2w)}{tan^3(2w)} dw=$

5) $\int \frac{\pi}{sin^2(5z)} dz$

Dr. Mohammed Jasim Mohammed Alfahdawi

Remark:

A. When the power of $\boldsymbol{sin(x)}$ or $\boldsymbol{cos(x)}$ is odd, we use:

$$sin^2(x) + cos^2(x) = 1$$

B. When the power of $\boldsymbol{sin(x)}$ or $\boldsymbol{cos(x)}$ is even, we use:

$$sin^2(x) = \frac{1}{2}(1 - cos2x) \textbf{ or } cos^2(x) = \frac{1}{2}(1 - cos2x)$$

Examples: Evaluate the following integrals:

1) $\int sin^3(x)dx$
$= \int \sin(x)\, sin^2(x)dx$
$= \int \sin(x)\left(1 - cos^2(x)\right)dx$
$= \int \sin(x) - \int cos^2(x)\sin(x)\, dx$
$= -\cos(x) - \frac{cos^3(x)}{3} + C$

---

2) $\int cos^2\,(y)dy$
$= \int \frac{1}{2}(1 + \cos(2y))dy$
$= \frac{1}{2}\int dy + \frac{1}{2}.\frac{1}{2}\int \cos(2y)\,.\,2dy$
$= \frac{1}{2}y + \frac{1}{4}\sin(2y) + C$

Dr. Mohammed Jasim Mohammed Alfahdawi

3) $\int cos^4(x)dx$

$$= \int (cos^2(x))^2 dx$$
$$= \int \left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$$
$$= \frac{1}{4}\int (1+2\cos(2x)+cos^2(2x))\,dx$$
$$= \frac{1}{4}\int 1dx + \frac{1}{4}\int 2\cos(2x)\,dx + \frac{1}{4}\int cos^2\,(2x)dx$$
$$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \frac{1}{2}(1+cos4x)dx$$
$$= \frac{1}{4}\int dx + \frac{1}{2}.\frac{1}{2}\int \cos(2x).2\,dx + \frac{1}{8}\int dx + \frac{1}{8}.\frac{1}{4}\int cos4x\,.4dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$$

Remark:

A. If the powers of ***tan***(***x***) and ***cot***(***x***) is **even**, we use:

$$tan^2(x) = sec^2(x) - 1$$

B. If the powers of ***sec***(***x***) and ***csc***(***x***) is **even**, we use:

$$sec^2(x) = 1 + tan^2(x)$$

Dr. Mohammed Jasim Mohammed Alfahdawi

Examples: Evaluate the following integrals:

1) $\int tan^2(3x)dx$

$= \int (sec^2(3x) - 1)dx$

$= \int sec^2(3x)dx - \int 1\,dx$

$= \frac{1}{3}\int sec^2(3x).3dx - \int dx$

$= \frac{1}{3}tan(3x) - x + C$

---

2) $\int sec^4(x)dx$

$= \int sec^2(x)sec^2(x)dx$

$= \int (1 + tan^2(x))\,sec^2(x)dx$

$= \int sec^2(x)dx + \int tan^2(x)\,sec^2(x)dx$

$= \tan(x) + \frac{tan^3(x)}{3} + C$

---

3) $\int csc^4(x)dx$

$= \int csc^2(x)csc^2(x)dx$

$= \int (1 + cot^2(x))\,csc^2(x)dx$

$= \int csc^2(x)dx + \int cot^2(x)\,csc^2(x)dx$

$= -\cot(x) - \frac{cot^3(x)}{3} + C$

H.W

Problems: Evaluate the following integrals:

1. $\int \sin(2t)\, dt$

2. $\int x cos(2x^2) dx$

3. $\int 2tan^2(2x) dx$

4. $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

5. $\int sin^2(x) \cos(x)\, dx$

6. $\int \frac{\sin(\frac{z-1}{3})}{cos^2(\frac{z-1}{3})} dz$

7. $\int \sin(t)\cos(t)(\sin(t) + \cos(t))\, dt$

Dr. Mohammed Jasim Mohammed Alfahdawi

Integrals of Inverse Trigonometric Functions:

We can derive all the integration forms from our derivatives forms as follows:

$$1)\ \frac{d}{du}\sin^{-1}(u)=\frac{1}{\sqrt{1-u^2}}\rightarrow\int\frac{1}{\sqrt{1-u^2}}du=\sin^{-1}(u)+C$$

$$2)\ \frac{d}{du}\cos^{-1}(u)=-\frac{1}{\sqrt{1-u^2}}\rightarrow\int\frac{1}{\sqrt{1-u^2}}du=-\cos^{-1}(u)+C$$

$$3)\ \frac{d}{du}\tan^{-1}(u)=\frac{1}{1+u^2}\rightarrow\int\frac{1}{1+u^2}du=\tan^{-1}(u)+C$$

$$4)\ \frac{d}{du}\cot^{-1}(u)=-\frac{-1}{1+u^2}\rightarrow\int\frac{-1}{1+u^2}du=-\cot^{-1}(u)+C$$

$$5)\ \frac{d}{du}\sec^{-1}(u)=\frac{1}{|u|\sqrt{u^2-1}}\rightarrow\int\frac{1}{|u|\sqrt{u^2-1}}du=\sec^{-1}(u)+C$$

$$6)\ \frac{d}{du}\csc^{-1}(u)=-\frac{-1}{|u|\sqrt{u^2-1}}\rightarrow\int\frac{-1}{|u|\sqrt{u^2-1}}du=-\csc^{-1}(u)+C$$

Examples: Evaluate the following integrals:

1) $\int\frac{dx}{\sqrt{1-4x^2}}$

$=\frac{1}{2}\int\frac{2dx}{\sqrt{1-(2x)^2}}$

$=\frac{1}{2}\sin^{-1}(2x)+C\quad or\quad\frac{-1}{2}\cos^{-1}(2x)+C$

---

2) $\int\frac{dt}{1+t^2}$

$=\tan^{-1}(t)+C\quad or\quad-\cot^{-1}(t)+C$

3) $\int\frac{dx}{x\sqrt{4x^2-1}}$

$=\frac{2dx}{2x\sqrt{2(2x)^2-1}}$

$$= sec^{-1}|2x| + C \quad or \quad -csc^{-1}|2x| + C$$

---

4) $\int \frac{-dx}{\sqrt{4-25x^2}}$

$$= \int \frac{-dx}{\sqrt{4\left(1-\frac{25}{4}x^2\right)}}$$

$$= \int \frac{-dx}{2\sqrt{\left(1-\left(\frac{5}{2}x\right)^2\right)}}$$

$$= \frac{-1}{2}.\frac{2}{5}\int \frac{\frac{5}{2}dx}{\sqrt{1-\left(\frac{5}{2}x\right)^2}}$$

$$= -\frac{1}{5}sin^{-1}\left(\frac{5}{2}x\right) + C \quad or \quad \frac{1}{5}cos^{-1}\left(\frac{5}{2}x\right) + C$$

---

5) $\int \frac{\cos(x)dx}{\sqrt{1-sin^2(x)}}$

$$= sin^{-1}(\sin(x)) + C$$

Dr. Mohammed Jasim Mohammed Alfahdawi

H.W

Problems: Evaluate the following integrals:

1- $\int \frac{-1}{\sqrt{1-16w^2}}dw$

2- $\int \frac{1}{1+25x^2}dx$

3- $\int \frac{1}{81u^2+9}du$

4- $\int \frac{\sin\left(\tan^{-1}(x)\right)}{1+x^2}dx$

5- $\int \frac{x}{9x+x^3}dx$

Integrals of Logarithmic Functions:

$$\because \frac{d}{du}\ln(u) = \frac{1}{u}du \rightarrow \int \frac{1}{u}du = ln|u| + C \quad , \qquad u \neq 0$$

Examples: Evaluate the following integrals:

1) $\int \frac{2}{x}dx = 2\int \frac{1}{x}dx = 2ln|x| + C$

2) $\int \left(\frac{3}{x^2} + \frac{5}{x}\right)dx$

$= 3\int x^{-2}dx + 5\int \frac{1}{x}dx$

$= 3\frac{x^{-1}}{-1} + 5ln|x| + C$

$= \frac{-3}{x} + 5ln|x| + C$

3) $\int \frac{x}{(2x^2+3)}dx$

$= \frac{1}{4}\int \frac{4x}{(2x^2+3)}dx$

$= \frac{1}{4}ln|2x^2 + 3| + C$

4) $\int \frac{\ln(x)}{x}dx$

$= \int \ln(x).\frac{1}{x}dx$

$= \frac{(\ln(x))^2}{2} + C$

5) $\int \frac{1}{xlnx}dx$

$= \int \frac{\frac{1}{x}}{\ln(x)}dx$

$= ln|\ln(x)| + C$

6) $\int \frac{e^x}{1+2e^x} dx$

$= \frac{1}{2} \int \frac{2e^x}{1+2e^x} dx$

$= \frac{1}{2} ln|1 + 2e^x| + C$

---

7) $\int \frac{\sec(2x)\tan(2x)}{\sec(2x)} dx$

$= \frac{1}{2} \int \frac{2\sec(2x)\tan(2x)}{\sec(2x)} dx$

$= ln|\sec(2x)| + C$

---

8) $\int \tan(u)\, du$

$= -\int \frac{-\sin(u)}{\cos(u)} du$

$= -ln|\cos(u)| + C$

Dr. Mohammed Jasim Mohammed Alfahdawi

9) $\int \sec(u)\, du$

$= \int \sec(u) . \frac{(\sec(u)+\tan(u))}{(\sec(u)+\tan(u))} du$

$= \int \frac{\sec^2(u)+\sec(u)\tan(u)}{\tan(u)+\sec(u)} du$

$$= \ln|\tan(u) + \sec(u)| + C$$

H.W

Problems: Evaluate the following integrals:

1. $\int \frac{1}{x-3} dx$
2. $\int \frac{xdx}{4x^2+1}$
3. $\int \frac{\sin(x)}{2-\cos(x)} dx$
4. $\int \frac{dx}{x.ln^5(x)}$
5. $\int \frac{x^{\frac{3}{2}}}{4x^{\frac{3}{2}}} + 6dx$

Dr. Mohammed Jasim Mohammed Alfahdawi

Integrals of Exponential Functions:

$$\because \frac{d}{du}e^{(u)} = e^{(u)}du \rightarrow \int e^{(u)}\,du = e^{(u)} + C \quad ,$$

Examples: Evaluate the following integrals:

1) $\int e^{2x}\,dx$

$= \frac{1}{2}\int 2e^{2x}\,dx$

$= \frac{1}{2}e^{2x} + C$

---

2) $\int e^{\sin 3x}.\cos 3x\,dx$

$= \frac{1}{3}\int 3.\,e^{\sin 3x}.\cos 3x\,dx$

$= \frac{1}{3}e^{\sin 3x} + C$

---

3) $\int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$

$Ln|e^{x} + e^{-x}| + C$

Dr. Mohammed Jasim Mohammed Alfahdawi

4) $\int e^{2x}\sin(e^{2x})dx$

$$= \frac{1}{2}\int 2.\, e^{2x} \sin(e^{2x}) dx$$

$$= -\frac{1}{2}\cos(e^{2x}) + C$$

---

5) $\int \frac{e^{sin^{-1}(2x)}}{\sqrt{1-4x^2}} dx$

$$= \frac{1}{2}\int \frac{2e^{sin^{-1}(2x)}}{\sqrt{1-4x^2}} dx$$

$$= \frac{1}{2} e^{sin^{-1}(2x)} + C$$

H.W

1. $\int \frac{e^x}{1+2e^x} dx$

2. $\int e^{\frac{x}{3}} \cos e^{\frac{x}{3}} dx$

3. $\int (e^{5x} cos\,(e^{5x}) - e^{\frac{x}{2}}) dx$

4. $\int \frac{e^{tan^{-1}(2t)}}{1+4t^2} dt$

Dr. Mohammed Jasim Mohammed Alfahdawi